A SHARP LOWER BOUND FOR THE CANONICAL VOLUME OF 3-FOLDS OF GENERAL TYPE

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ABSTRACT. Let V be a smooth projective 3-fold of general type. Denote by K^3, a rational number, the self-intersection of the canonical sheaf of any minimal model of V. One defines K^3 as the canonical volume of V. Assume $p_g(V) \geq 2$. We show that $K^3 \geq \frac{1}{3}$, which is a sharp lower bound. Then we classify those V with small volume. We also give some new examples with $p_g = 2$ which have maximal canonical stability index. Finally, we give an application to 4-folds of general type.

1. Introduction

To classify algebraic varieties is one of the main goals of algebraic geometry. For a long time, we have been interested in classifying algebraic 3-folds of general type which are, naturally, quite important objects for birationalists. The book [9] edited by Corti and Reid explains some ways to understand the explicit structure of algebraic threefolds. One might have noted, however, that this is a very big topic and that even there is not any answer to lots of very elementary questions.

Let V be a smooth 3-dimensional projective variety of general type. According to Mori's Minimal Model Program (see for instance a sample of references [18, 21, 27]), V has at least one minimal model X which is normal projective with only $\mathbb{Q}$-factorial terminal singularities. Denote by $K^3 := K_X^3$, which is a positive rational number. Reid (the last section of [27]) first showed that all minimal models of V are in fact isomorphic in codimension 1. One also knows the uniqueness of the Zariski decomposition for canonical bundles mK_V by Kawamata (see page 355 of [18] and Proposition 4 in [17]). Therefore one can see that the canonical volume K^3 is uniquely determined by the birational equivalence class of V. K^3 is an important invariant and the value $\mathrm{Vol}(V) := K^3$ is referred to as *the canonical volume of V*. Obviously V has some other important birational invariants such as the geometric genus $p_g(V) := \dim H^0(V, \Omega_V^3)$, the irregularity $q(V) := \dim H^1(V, \mathcal{O}_V)$, the second irregularity $h^2(\mathcal{O}_V) := \dim H^2(V, \mathcal{O}_V)$. These invariants determine the holomorphic Euler-Poincaré characteristic

$$\chi(\mathcal{O}_V) := 1 - q(V) + h^2(\mathcal{O}_V) - p_g(V).$$

A difficulty arises in the study of 3-folds of general type is that K^3 is only a rational number, rather than an integer. Furthermore it may be, in fact, very small. For example, among known ones by Fletcher-Reid (page 151 in [9]), $\text{Vol}(V)$ could be as small as $\frac{1}{420}$. Inspired by a series of lectures of Y.T. Siu, we consider the following very natural and interesting question.

Question 1.1. What is the sharp lower bound of the volume among all minimal projective 3-folds of general type?

Since several decades, it is a fact that the treatment of pluricanonical maps is easier when K^3 is larger. Morever, the treatment of pluricanonical maps is much easier when $p_g > 0$. This is due to the fact that under this assumption there are no gaps, since a pluricanonical system $|(m+1)K|$ always contains the pluricanonical system $|mK|$ as a subsystem. From this point of view, the lower bound for K^3 becomes quite important. After the first manuscript of this paper was finished, we noted that Hajime Tsuji, Christopher D. Hacon and James Mckernan [29, 14] announced the existence of an uniform lower bound of K^3. In this paper, we are only able to answer Question 1.1 under the extra assumption $p_g \geq 2$. We will see that the volume strongly affects the canonical stability index (see 1.2 below for the definition) of the given 3-fold V.

Definition 1.2. The *canonical stability index* $n_0(Y)$ of any smooth projective variety Y of general type is defined to be the minimal integer n_0 such that for all $i \geq n_0$ the i-th pluricanonical map is birational onto its image. For $p_g > 0$, it is simply the smallest integer n_0 such that the n_0-th pluricanonical map is birational onto its image.

1.3. Upper bound for $n_0(V)$. Let V be a smooth 3-dimensional projective variety of general type. In [6], the following was proved:
  (1) $n_0(V) \leq 5$ whenever $p_g(V) \geq 4$ and this bound is sharp;
  (2) $n_0(V) \leq 6$ whenever $p_g(V) = 3$ and this bound is sharp;
  (3) $n_0(V) \leq 8$ whenever $p_g(V) = 2$.
  Recently I was informed of separately by both Ezio Stagnaro and Christopher Hacon [13] that the bound of $n_0(V)$ in 1.3(3) is also sharp. As being stated in Theorem 1.4(2), more examples with $p_g = 2$ and with maximal canonical stability index can be found if the canonical volume satisfies certain inequality.

My main results are as follows.

Theorem 1.4. *Let V be a smooth 3-dimensional projective variety of general type. Assume $p_g(V) \geq 2$. Then*
  (1) $K^3 \geq \frac{1}{3}$ and this bound is sharp.
  (2) If $\frac{1}{3} \leq K^3 < \frac{5}{14}$, then $p_g(V) = 2$, $n_0(V) = 8$, V is canonically fibred by surfaces with $(c_1^2, p_g) = (1, 2)$, and the 7-canonical map of V is generically finite of degree 2.

The proof of the main theorem has interesting applications to the effect that we are able to classify some classes of 3-folds up to explicit structures. For instance, we have

Theorem 1.5. *Let V be a smooth 3-dimensional projective variety of general type. The following is true:*

(1) if $p_g(V) \geq 3$, then $K^3 \geq 1$ and this bound is sharp;

(2) if $p_g(V) \geq 4$, then $K^3 \geq 2$ and this bound is sharp;

(3) if $K^3 < \frac{1}{2}$ and $p_g(V) = 2$, then V is canonically fibred by surfaces with $c_1^2 = 1$. In particular, $q(V) = 0$ and $h^2(\mathcal{O}_V) \leq 1$;

(4) if $K^3 < \frac{4}{3}$ and $p_g(V) = 3$, then V is canonically fibred by curves of genus 2 over a birationally ruled surface and the 4-canonical map is generically finite of degree 2.

(5) if $K^3 < \frac{9}{4}$ and $p_g(V) = 4$, then V is either a double cover over $\mathbb{P}^3$ or canonically fibred by curves of genus 2 over a ruled surface.

All the above statements have supporting examples. More detailed classification to those V with $p_g = 2$ and 3 is given in section 2 and section 3. The method works for the case $p_g \leq 1$ for which we omit the effective result believing that it might be far from sharp. Instead we present effective results on 4-folds in the last section.

Theorem 1.6. *Let Y be a minimal (i.e. K_Y being nef) projective 4-fold of general type with only canonical singularities. Assume $p_g(Y) \geq 2$ and that Y is not canonically fibred by 3-folds of geometric genus 1. Then $n_0(Y) \leq 24$. Furthermore $K_Y^4 \geq \frac{1}{81}$ if Y is not canonically fibred by any irrational pencil of 3-folds either.*

Notations

K^3	the canonical volume of a 3-fold in question
p_g	the geometric genus
$q(V)$	the irregularity of V
$h^2(\mathcal{O}_V)$	the second irregularity of a 3-fold V
$\chi(\mathcal{O})$	the Euler Poincare characteristic
(c_1^2, p_g)	invariants of a minimal surface of general type
$b := g(B)$	the genus of a curve B
$=_{\mathbb{Q}}$	$\mathbb{Q}$-linear equivalence
$\equiv$	numerical equivalence
$\sim$	linear equivalence
$\Phi_{|L|}$	the rational map corresponding to the linear system $|L|$
$\ulcorner . \urcorner$	the round up of a $\mathbb{Q}$-divisor
φ_m	the m-th pluricanonical map
$n_0(Y)$	the canonical stability index of Y
$D|_S$	the restriction of the divisor D to S
$|D||_S$	the restriction of the linear system $|D|$ to S

$P_m(V)$	the m-th plurigenus of V
$D \cdot C$	the intersection number between a divisor D and curve C
$\pi : X' \to X$	a smooth birational modification
$f : X' \to B$	an induced fibration from φ_1
M	the movable part of $|K_{X'}|$
Z	the fixed part of $|K_{X'}|$
S	a generic irreducible element of $|M|$
$\sigma : S \to S_0$	contraction onto the minimal model
$|G|$	a movable linear system on S
C	a generic irreducible element of $|G|$
ξ	the intersection number $\pi^*(K_X) \cdot C$ on X'
β	a positive real number such that $\pi^*(K_X)|_S - \beta C$ is pseudo effective
M_i	the movable part of $|iK_{X'}|$
S_2	a generic irreducible element of $|M_2|$
L_2	the divisor $M_2|_{S_2}$ on S_2

2. The case $p_g(V) = 2$

2.1. Setting. Given a smooth projective 3-dimensional variety of general type with $p_g(V) \geq 2$, let us consider a minimal model as explained in the introduction. So we assume that X is a minimal model of V with at worst $\mathbb{Q}$-factorial terminal singularities. We know $p_g(X) = p_g(V)$. Take a birational modification $\pi : X' \to X$, which exists by Hironaka's theorem, such that

  (i) X' is smooth;

  (ii) the movable part of $|K_{X'}|$ is base point free.

  (iii) $\pi^*(K_X)$ can be written as an effective $\mathbb{Q}$-divisor with normal crossings.

Denote by g the composition $\varphi_1 \circ \pi$. So $g : X' \longrightarrow W \subseteq \mathbb{P}^{p_g(X)-1}$ is a morphism. Let $g : X' \xrightarrow{f} B \xrightarrow{s} W$ be the Stein factorization of g. So we have the following commutative diagram:

$$
\begin{array}{ccc}
X' & \xrightarrow{f} & B \\
\pi \downarrow & & \downarrow s \\
X & \xrightarrow[\varphi_1]{} & W
\end{array}
$$

  Write

$$K_{X'} =_{\mathbb{Q}} \pi^*(K_X) + E = M + Z,$$

where M is the movable part of $|K_{X'}|$, Z the fixed part and E an effective $\mathbb{Q}$-divisor which is a $\mathbb{Q}$-sum of distinct exceptional divisors. Throughout we always mean $\pi^*(K_X)$ by $K_{X'} - E$. So, whenever we

take the round up of $m\pi^*(K_X)$, we always have $\lceil m\pi^*(K_X) \rceil \le mK_{X'}$ for all positive number m. We may also write

$$\pi^*(K_X) =_{\mathbb{Q}} M + E',$$

where $E' = Z - E$ is actually an effective $\mathbb{Q}$-divisor.

When $\dim\varphi_1(X) = 2$, we see that a general fiber of f is a smooth projective curve C of genus $g \ge 2$. When $\dim\varphi_1(X) = 1$, we may see from the subadditivity (1.7) in Mori's paper [23] that a general fiber S of f is a smooth projective surface S of general type. The invariants of S are $(c_1^2, p_g) = (c_1^2(S_0), p_g(S))$ where S_0 is the minimal model of S and $\sigma : S \longrightarrow S_0$ the contraction.

We always mean *a generic irreducible element S of $|M|$* by either a general member of $|M|$ whenever $\dim\varphi_1(X) \ge 2$ or, otherwise, a general fiber of f.

Definition 2.2. By abuse of concepts, we will also define *a generic irreducible element S* of an arbitrary linear system $|M|$ on a general variety V in a similar way. Assume that $|M|$ is movable. A generic irreducible element S is defined to be a generic irreducible component in a general member of $|M|$. So if $|M|$ is composed with a pencil (i.e. $\dim \Phi_{|M|}(V) = 1$), $S \le M$ and $M \equiv tS$ for some integer $t \ge 1$. Clearly it may happen that sometimes $S \not\sim M$.

We need the following lemma in the proof.

Lemma 2.3. *Keep the same notation as above. Let X be a minimal 3-fold of general type with at worst $\mathbb{Q}$-factorial terminal singularities. Let $\pi : X' \to X$ be the same modification as above. Suppose we are in the situation $\dim\varphi_1(X) = 1$. Let $f : X' \longrightarrow B$ be an induced fibration. Suppose $g(B) > 0$. Then X has another minimal model Y such that there is a fibration $f_Y : Y \longrightarrow B$ which is induced by the canonical pencil $|K_Y|$. In particular, the movable part of $|K_Y|$ is base point free. (Therefore, sometimes, we will replace X by Y and we simply study on Y.)*

Proof. According to the MMP (see [18, 21, 27]), we may take a relatively minimal model $f_Y : Y \longrightarrow B$ of f such that K_Y is f_Y-nef and that Y is $\mathbb{Q}$-factorial with terminal singularities. According to Theorem 1.4 of [25], $K_{Y/B}$ is nef. (This is a direct consequence of the well-known semi-positivity of higher direct image of dualizing sheaves by Fujita, Kawamata, Kollár, Nakayama and Viehweg.) Because $g(B) > 0$, K_Y is nef. Thus Y is a minimal model of X'. Take a common smooth birational modification X'' over both X' and Y. We then have the

following commutative diagram:

$$
\begin{array}{ccc}
X'' & \xrightarrow{\ \theta\ } & Y \\
\theta' \downarrow & & \downarrow f_Y \\
X' & \xrightarrow{\ \ f\ \ } & B
\end{array}
$$

We know that the movable part of $K_{X''}$ is base point free. Because $p_g(X'') = p_g(Y)$ and θ is a birational morphism, we see that the push forward of the movable part of $|K_{X''}|$ onto Y is the movable part of $|K_Y|$. We are done. $\qquad\square$

2.4. A known theorem. In order to frequently apply it, we rephrase a very effective method (Theorem 2.2 in [6]) on how to estimate certain intersection numbers on X as follows:

> Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities and assume $p_g(X) \geq 2$. Keep the same notations as in 2.1. Pick up a generic irreducible element S of $|M|$. Suppose that, on the smooth surface S, there is a movable linear system $|G|$ and set C to be a generic irreducible element of $|G|$. Denote $\xi := (\pi^*(K_X) \cdot C)_{X'}$ and
>
> $$
> p := \begin{cases} 1 & \text{if } \dim \varphi_1(X) \geq 2 \\ p_g(X) - 1 & \text{otherwise.} \end{cases}
> $$
>
> Assume
> • (i) there is a positive integer m such that the linear system
> $$
> |K_S + \lceil (m-2)\pi^*(K_X)|_S \rceil|
> $$
> (of which the corresponding rational map is denoted by Φ) separates different generic irreducible elements of $|G|$. Namely, if C_1 and C_2 are different generic irreducible elements of $|G|$, then $\overline{\Phi(C_1)} \neq \overline{\Phi(C_2)}$;
> • (ii) there is a rational number $\beta > 0$ such that $\pi^*(K_X)|_S - \beta C$ is numerically equivalent to an effective $\mathbb{Q}$-divisor;
> • (iii) either the inequality
> $$
> \alpha_0 := (\lceil (m-1-\tfrac{1}{p}-\tfrac{1}{\beta})\pi^*(K_X)|_S \rceil \cdot C)_S \geq 2
> $$
> or at least $\alpha := (m-1-\tfrac{1}{p}-\tfrac{1}{\beta})\xi > 1$ holds (observe in fact that $\alpha_0 \geq \alpha$).
>
> Then we have the inequality $m\xi \geq 2g(C) - 2 + \alpha_0$. Furthermore, φ_m of X is birational onto its image provided either $\alpha > 2$ or $\alpha_0 = 2$ and C is non-hyperelliptic.

Remark 2.5. As far as the situation is involved, we don't have to verify the condition (i) in this paper since that was already done in [6]. The main reason is that we are treating the case with $p_g(X) > 0$. On how to take a β satisfying the assumption 2.4(ii), please refer to 2.8 below.

2.6. Base point freeness of $|2K|$**.** According to Bombieri [2], Reider [28], Catanese-Ciliberto [5] and P. Francia [11] (or directly referring to Theorem 3.1 in the survey article by Ciliberto [8]), the bicanonical system $|2K|$ of a minimal surface of general type with $p_g > 0$ is base point free. We will frequently apply this important result in the context.

2.7. Numerical type of a surface of general type. Given a smooth projective surface S of general type, we denote by $\sigma : S \longrightarrow S_0$ the contraction onto the minimal model. For the need of our proof and according to the standard surface theory (see [1] and [2]), we classify S into the five numerical classes as follows:

(1) S is of type $(1,1)$ if $K_{S_0}^2 = 1$ and $p_g(S) = 1$;

(2) S is of type $(1,2)$ if $K_{S_0}^2 = 1$ and $p_g(S) = 2$;

(3) S is of type $(2,3)$ if $K_{S_0}^2 = 2$ and $p_g(S) = 3$;

(4) S is of type 2^+ if $K_{S_0}^2 \geq 2$ but S is not of type $(2,3)$ (all surfaces with $p_g \geq 3$ fall into this type because of the Noether inequality: $K^2 \geq 2p_g - 4$);

(5) S is of type $(1,0)$ if $K_{S_0}^2 = 1$ and $p_g(S) = 0$.

Because $p_g(V) \geq 2$, we see that $p_g(S) > 0$. So type $(1,0)$ never happens under the assumption $p_g(V) \geq 2$.

2.8. How to take a suitable β**?** We keep the notations in both 2.1 and 2.4. Assume $p_g(X) \geq 2$, $\dim \varphi_1(X) = 1$ and $b = g(B) = 0$. Let S be a general fiber of the induced fibration $f : X' \longrightarrow B$. Denote by $\sigma : S \longrightarrow S_0$ the contraction onto the minimal model. By section 4 (at page 526 and page 527) of [6], we may always choose a sequence of rational numbers $\beta_0 \mapsto \frac{1}{2}$ with $\beta_0 < \frac{1}{2}$ such that $\pi^*(K_X)|_S - \beta_0 \sigma^*(K_{S_0})$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. So we may take a suitable β accordingly:

• If $p_g(S) > 0$ and take $G := 2\sigma^*(K_{S_0})$, we may choose $\beta = \frac{1}{2}\beta_0$;

• If S is of type $(1,2)$ and take G to be the movable part of $|\sigma^*(K_{S_0})|$, we may choose $\beta = \beta_0$;

• If S is of type $(2,3)$ and take G to be the movable part of $|\sigma^*(K_{S_0})|$, we may choose $\beta = \beta_0$.

Usually β can be larger whenever $p_g(X)$ is larger. This is clear by virtue of the method in [6]. For example, if $p_g(X) \geq 3$, we can find a larger β as we will do in the next section.

2.9. Set up for the case $p_g(V) = 2$. From now on within this section, we assume $p_g(V) = 2$. Then B is a smooth curve. Let S be a general fiber of f. One may always write $M = \sum_{i=1}^{a_1} S_i$ where all those $S_i's$ are disjoint from each other and $a_1 \geq p_g(V) - 1$. So we have

$$(2.1) \qquad K_X^3 = (\pi^*(K_X))^3 = a_1 \pi^*(K_X)^2 \cdot S + \pi^*(K_X)^2 \cdot E'.$$

Assume $b = g(B) = 0$. We have $p = 1$ by definition. On the other hand, we set the divisor G on S as follows:

$$G := \begin{cases} \text{the movable part } C \text{ of } |\sigma^*(K_{S_0})|, & \text{if } S \text{ is of type } (1,2) \text{ or } (2,3) \\ 2\sigma^*(K_{S_0}), & \text{if } S \text{ is of type } 2^+ \text{ or } (1,1). \end{cases}$$

Also set

$$\xi := (\pi^*(K_X)|_S \cdot G)_S = (\pi^*(K_X) \cdot G)_{X'} = (K_X \cdot \pi_*(G))_X$$

which is independent of the modification π by the intersection theory. All these settings are to prepare for estimating ξ by means of the technique 2.4.

2.10. The case $b > 0$. Recall the nefness of K_X. This case is simple since the movable part of $|K_X|$ is already base point free (see Lemma 2.3 for a possible replacement of X). So one has

$$\pi^*(K_X)|_S \sim \sigma^*(K_{S_0})$$

and $a_1 \geq p_g(V) = 2$. Thus we have

$$K_X^3 \geq 2\sigma^*(K_{S_0})^2 \geq 2.$$

Next we only have to study the case $b = 0$.

2.11. The type 2^+. Because $\pi^*(K_X)$ is nef and big, the equation (2.1) in 2.9 gives $K_X^3 \geq \pi^*(K_X)^2 \cdot S$. So the main point is to estimate $\pi^*(K_X)^2 \cdot S$. By 2.8, one may find a sequence of rational numbers $\beta > 0$ with $\beta \to \frac{1}{4}$ and $\beta < \frac{1}{4}$ such that $\pi^*(K_X)|_S - \beta G$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. Thus we have

$$\pi^*(K_X)^2 \cdot S \geq \beta \xi.$$

We are reduced to estimate the rational number ξ. Let C be a generic irreducible element of $|G|$. We know that $|G|$ is base point free. So C is a smooth curve with $\deg(K_C) \geq 12$. Note that C must be non-hyperelliptic since $n_0(S) \leq 3$ by Bombieri [2] and Reider [28].

Take $m_1 = 7$. Then $\alpha = (m_1 - 1 - \frac{1}{p} - \frac{1}{\beta})\xi > 0$. So $\alpha_0 > 0$. Because in this case G is an even divisor by definition, we must have $\alpha_0 \geq 2$. So 2.4 gives $\xi \geq 2$. Thus we have $K_X^3 \geq \frac{1}{2}$.

2.12. The type $(2,3)$. Let C be a generic irreducible element of $|G|$. Then C is a curve of genus 3 and $C^2 = K_{S_0}^2 = 2$ (see pages 226-227 of

[1]). According to 2.8, we may take a rational number $\beta \mapsto \frac{1}{2}$ such that $\pi^*(K_X)|_S - \beta C$ is pseudo-effective. Thus $\xi \geq \beta C^2 \geq 1$. So we have

$$K_X^3 \geq \pi^*(K_X)^2 \cdot S \geq \beta\xi \geq \frac{1}{2}.$$

2.13. The type $(1,1)$**.** This situation has to be studied in an alternative way. We have an induced fibration $f : X' \longrightarrow B$. Because $|K_{X'}|$ is composed with a pencil, $b = 0$ and $f_*\omega_{X'}$ is a line bundle in this case, we see that $\deg f_*\omega_{X'} > 0$ by the Riemann-Roch theorem. According to Bombieri [2], a surface S of type (1,1) has $q(S) = 0$. So $R^1 f_*\omega_{X'} = 0$. Applying Kollár's formulae (Proposition 7.6 at page 36 of [20]), one gets $h^2(\mathcal{O}_X) = h^1(f_*\omega_{X'}) = 0$ and $q(X) = 0$. Thus $\chi(\mathcal{O}_X) = -1$. By the plurigenus formula of Reid (see Chapter III of [26]) and omitting the correction term, we have

$$P_2(X) \geq \frac{1}{2}K_X^3 - 3\chi(\mathcal{O}_X) > 3.$$

This means $P_2(X) \geq 4$ because $K_X^3 > 0$. Let M_2 be the movable part of $|2K_{X'}|$. We consider the natural restriction map γ:

$$H^0(X', M_2) \xrightarrow{\gamma} U_2 \subset H^0(S, M_2|_S) \subset H^0(S, 2K_S),$$

where U_2 is the image of γ as a $\mathbb{C}$-subspace of $H^0(S, M_2|_S)$. Because $h^0(2K_S) = K^2 + \chi(\mathcal{O}_S) = 3$, we see that $1 \leq \dim_{\mathbb{C}} U_2 \leq 3$. Denote by Λ_2 the linear system corresponding to U_2. We have $\dim \Lambda_2 = \dim_{\mathbb{C}} U_2 - 1$.

Case 1. $\dim_{\mathbb{C}} U_2 = 3$.

Since Λ_2 is a sub-system of $|2K_S|$, we see that the restriction of $\phi_{2,X'}$ to S is exactly the bicanonical map of S. Because $\phi_{2,S}$ is a generically finite morphism of degree 4, $\phi_{2,X'}$ is also a generically finite map of degree 4. (Here let me slightly explain the fact about $\phi_{2,S}$. As I have explained in 2.11, any minimal surface of general type with $p_g > 0$ has a base point free bicanonical system. It follows directly that the movable part of $|2K_S|$ defines a generically finite morphism of degree 4. Suitable references might be [1] and [2].)

Let $S_2 \in |M_2|$ be a general member. We can further remodify π such that $|M_2|$ is base point free too. Then S_2 is a smooth projective irreducible surface of general type. On the surface S_2, denote $L_2 := S_2|_{S_2}$. L_2 is a nef and big divisor. We have

$$2\pi^*(K_X)|_{S_2} \geq S_2|_{S_2} = L_2.$$

We consider the natural map

$$H^0(X', S_2) \xrightarrow{\gamma'} \overline{U_2} \subset H^0(S_2, L_2),$$

where $\overline{U_2}$ is the image of γ'. Denote by $\overline{\Lambda_2}$ the linear system corresponding to $\overline{U_2}$. Because φ_2 is a generically finite map of degree 4, we

see that $|L_2|$ has a sub-system $\overline{\Lambda}_2$ which gives a generically finite map of degree 4. By direct calculation on the surface S_2, we have

$$L_2^2 \geq 4(\dim_{\mathbb{C}} \overline{\Lambda}_2 - 1) \geq 4(P_2(X) - 3) \geq 4.$$

Therefore we have

$$K_X^3 \geq \frac{1}{8}L_2^2 \geq \frac{1}{2}.$$

Case 2. $\dim_{\mathbb{C}} U_2 = 2$.

In this case, $\dim \varphi_2(S) = 1$ and $\dim \varphi_2(X) = 2$. We may further remodify π such that $|M_2|$ is base point free. Taking the Stein factorization of $\Phi_{2K_{X'}}$, we get a induced fibration $f_2 : X' \longrightarrow B_2$ where B_2 is a surface. Let C be a general fiber of f_2. We see that S is naturally fibred by curves with the same numerical type as C. On the surface S, we have a free pencil $\Lambda_2 \subset |2K_S|$ (here we mean the movable part of Λ_2 has no base points). Let $|C_0|$ be the movable part of Λ_2. Then $h^0(S, C_0) = 2$ because $\dim_{\mathbb{C}} U_2 = 2$ in this case. Because $q(S) = 0$ ([1, 2]), we see that $|C_0|$ is a rational pencil. So a general member of $|C_0|$ is an irreducible curve with the same numerical type with C.

Claim 2.14. *Let S be a surface of type (1,1). Suppose that there is an effective irreducible curve C on S such that $C \leq \sigma^*(2K_{S_0})$ and $h^0(S, C) = 2$. Then $C \cdot \sigma^*(K_{S_0}) \geq 2$.*

Proof. We may assume that $|C|$ is base point free. Otherwise, we blow-up S at base points of $|C|$ and consider the movable part. Denote $C_1 := \sigma(C)$. Then $h^0(S_0, C_1) \geq 2$. Suppose $C \cdot \sigma^*(K_{S_0}) = 1$. Then $C_1 \cdot K_{S_0} = 1$. Because $p_a(C_1) \geq 2$, we see that $C_1^2 > 0$. From $K_{S_0}(K_{S_0} - C_1) = 0$, we get $(K_{S_0} - C_1)^2 \leq 0$, i.e. $C_1^2 \leq 1$. Thus $C_1^2 = 1$ and $K_{S_0} \equiv C_1$. This means $K_{S_0} \sim C_1$ by virtue of [1, 2, 3], which is impossible because $p_g(S) = 1$. So $C \cdot \sigma^*(K_{S_0}) \geq 2$ □

According to Claim 2.14, we have $(C_0 \cdot \sigma^*(K_{S_0}))_S \geq 2$. We also have

$$(\pi^*(K_X) \cdot C)_{X'} = (\pi^*(K_X)|_S \cdot C_0)_S.$$

According to 2.8, there is a rational number $\beta_0 \mapsto \frac{1}{2}$ such that $\pi^*(K_X)|_S - \beta_0\sigma^*(K_{S_0})$ is pseudo-effective. Thus $(\pi^*(K_X)|_S \cdot C_0)_S \geq \frac{1}{2}\sigma^*(K_{S_0}) \cdot C_0 \geq 1$. Now we study on the surface S_2. We may write

$$S_2|_{S_2} \sim \sum_{i=1}^{a_2} C_i \equiv a_2 C,$$

where the $C_i's$ are fibers of f_2 and $a_2 \geq P_2(X) - 2 \geq 2$. Noting that $2\pi^*(K_X)|_{S_2} \geq S_2|_{S_2}$, we get

$$4K_X^3 \geq 2\pi^*(K_X)^2 \cdot S_2 = 2(\pi^*(K_X)|_{S_2})_{S_2}^2$$
$$\geq 2(\pi^*(K_X)|_{S_2} \cdot C)_{S_2}$$
$$= 2(\pi^*(K_X) \cdot C)_{X'} \geq 2.$$

So we still have $K_X^3 \geq \frac{1}{2}$.

Case 3. $\dim_{\mathbb{C}} U_2 = 1$.

In this case, $\dim \varphi_2(X) = 1$. Because $p_g(X) > 0$, we see that both φ_2 and φ_1 induce the same fibration $f : X' \longrightarrow B$ after taking the Stein factorization of them. So we may write

$$2\pi^*(K_X) \sim \sum_{i=1}^{a_2'} S_i + E_2' \equiv a_2'S + E_2',$$

where the $S_i's$ are fibers of f, E_2' is an effective $\mathbb{Q}$-divisor, $a_2' \geq P_2(X) - 1 \geq 3$ and S is a surface of type $(1,1)$. So we get

$$2K_X^3 \geq 3(\pi^*(K_X)|_S)_S^2 \geq 3\beta_0 \pi^*(K_X)|_S \cdot \sigma^*(K_{S_0})$$

where $\beta_0 \mapsto \frac{1}{2}$.

Now we may apply 2.4 to estimate $\pi^*(K_X)|_S \cdot \sigma^*(K_{S_0})$. We have $p = 1$ and set $G := 2\sigma^*(K_{S_0})$. 2.8 allows us to take $\beta = \frac{1}{2}\beta_0$. Note that a generic irreducible element $C \in |G|$ on S is a non-hyperelliptic curve (since we know $n_0(S) \leq 3$ by Bombieri [2]) and $\deg(K_C) = 6$. Because C is actually an even divisor on the surface S, the number α_0 in 2.4 is a positive even number whenever $l = m - 1 - \frac{1}{p} - \frac{1}{\beta} > 0$ for some integer m. This means we have $\alpha_0 \geq 2$ whenever $l > 0$. Now taking $m_1 = 7$, we may verify $l > 0$ and so all assumptions in 2.4 are satisfied. Then 2.4 gives $\xi := \pi^*(K_X)|_S \cdot G \geq \frac{8}{7}$. Take $m_2 = 8$. Then similarly we have $\xi \geq \frac{5}{4}$. We may check that this is the best bound we could get from this technique. Thus we have $\pi^*(K_X)|_S \cdot \sigma^*(K_{S_0}) \geq \frac{5}{8}$ and so $K_X^3 \geq \frac{15}{32}$.

Thus we have $K_X^3 \geq \frac{15}{32}$.

2.15. The type $(1, 2)$. According to 2.8, we may take $\beta \mapsto \frac{1}{2}$. From section 4 (page 527) of [6], we know $\xi \geq \frac{3}{5}$. Take $m_1 = 8$. Then $\alpha = (m_1 - 1 - 1 - \frac{1}{\beta})\xi > 2$. So 2.4 gives $\xi \geq \frac{5}{8}$. Take $m_2 = 9$. Then $\alpha = (m_2 - 2 - \frac{1}{\beta})\xi > 3$. 2.4 gives $\xi \geq \frac{2}{3}$. One may check that this is the best we could get. Thus we have

$$K_X^3 \geq \pi^*(K_X)^2 \cdot S \geq \beta\xi \geq \frac{1}{3}.$$

We go on studying this case from the point of view of the canonical stability index.

Claim 2.16. *Under the situation 2.15, $n_0(V) = 8$ if and only if $\xi = \frac{2}{3}$. In particular $n_0(V) = 8$ whenever $K^3 < \frac{5}{14}$.*

Proof. If $\xi > \frac{2}{3}$, let us take $m = 7$. Then $\alpha = (m - 2 - \frac{1}{\beta})\xi > 2$. 2.4 says that φ_7 is birational and $\xi \geq \frac{5}{7}$. So we get $K_X^3 \geq \frac{5}{14}$. This means that $n_0(V) = 8$ implies $\xi = \frac{2}{3}$.

To the contrary, assume $\xi = \frac{2}{3}$, we want to show that $n_0(V) = 8$ and that the 7-canonical map is generically finite of degree 2.

We consider the sub-system

$$|K_{X'} + \ulcorner 5\pi^*(K_X) \urcorner + S| \subset |7K_{X'}|.$$

This system obviously separates different generic irreducible elements of $|M|$ because $K_{X'} + \ulcorner 5\pi^*(K_X) \urcorner + S \geq S$ and $|S|$ is composed with a rational pencil. By the Tankeev principle for birationality, it is sufficient to prove that $|7K_{X'}||_S$ gives a birational map. Noting that $5\pi^*(K_X)$ is nef and big, the Kawamata-Viehweg vanishing theorem (see [10, 15, 30]) gives the surjective map

$$H^0(X', K_{X'} + \ulcorner 5\pi^*(K_X) \urcorner + S)$$
$$\longrightarrow \quad H^0(S, K_S + \ulcorner 5\pi^*(K_X) \urcorner|_S).$$

We are reduced to prove that $|K_S + \ulcorner 5\pi^*(K_X) \urcorner|_S|$ gives a birational map. We still apply the Tankeev principle. Because

$$K_S + \ulcorner 5\pi^*(K_X) \urcorner|_S \geq K_S + \ulcorner 5\pi^*(K_X)|_S \urcorner,$$

the linear system $|K_S + \ulcorner 5\pi^*(K_X) \urcorner|_S|$ separates different irreducible elements of $|G|$ where G is defined as in 2.9. This is because $\ulcorner 5\pi^*(K_X) \urcorner|_S \geq 0$ and $K_S \geq C$ and $|C|$ is composed with a rational pencil of curves. Note also here that $q(S) = 0$ and C is a smooth curve of genus 2 (see page 225 of [1]). Now pick up a generic irreducible element $C \in |G|$. By 2.8 or [6], there is a rational number $\beta \mapsto \frac{1}{2}$ and an effective $\mathbb{Q}$-divisor H on S such that

$$\frac{1}{\beta}\pi^*(K_X)|_S \equiv C + H.$$

By the vanishing theorem, we have the surjective map

$$H^0(S, K_S + \ulcorner 5\pi^*(K_X)|_S - H \urcorner) \longrightarrow H^0(C, D),$$

where $D := \ulcorner 5\pi^*(K_X)|_S - C - H \urcorner|_C$ is a divisor on C. Noting that

$$5\pi^*(K_X)|_S - C - H \equiv (5 - \frac{1}{\beta})\pi^*(K_X)|_S$$

and that C is nef on S, we have $\deg(D) \geq \alpha$ and thus $\deg(D) \geq \alpha_0$.

Now if $\xi = \frac{2}{3}$, then $\deg(D) \geq 2$. This means $|K_C + D|$ is base point free. Noting that C is a curve of genus 2, $|K_C + D|$ gives a finite map of degree ≤ 2. Thus φ_7 must be a generically finite map if it is not birational.

Now we show that the 7-canonical map is not birational. Denote by M_7 the movable part of $|7K_{X'}|$ and by M_7' the movable part of $|K_{X'} + \ulcorner 5\pi^*(K_X) \urcorner + S|$. Apparently, one has

$$M_7'|_S \leq M_7|_S \leq 7\pi^*(K_X)|_S.$$

Because $7\pi^*(K_X)|_S \cdot C = \frac{14}{3} < 5$, we have $M_7|_S \cdot C \le 4$. On the other hand, we have $M_7'|_S \cdot C \ge \deg(K_C + D) \ge 4$. This means that $\deg(K_C + D) = 4$. It is obvious that

$$\lceil 7\pi^*(K_X)|_S \rceil|_C \sim 5P$$

where P is a point of C such that $\mathcal{O}_C(2P) \cong \omega_C$. Furthermore one sees that $(\lfloor 7\pi^*(K_X) \rfloor|_S)|_C \sim nP$ with $n \le 4$. Because $M_7 \le \lfloor 7\pi^*(K_X) \rfloor$, we must have $n = 4$ and $K_C + D \sim 4P$. So $|K_C + D|$ must give a finite map of degree 2. So is φ_7. $\square$

Corollary 2.17. *Assume $p_g(V) = 2$. If $K^3 < \frac{5}{14}$, then V is of type $(1, 2)$ and $n_0(V) = 8$.*

Proof. This is clear by virtue of 2.10, 2.11, 2.12, 2.13, 2.15 and 2.16. $\square$

Corollary 2.18. *Assume $p_g(V) = 2$. If $K^3 < \frac{1}{2}$, then V is canonically fibred by surfaces of type $(1, 1)$ or $(1, 2)$. In particular, $q(V) = 0$ and $h^2(\mathcal{O}_V) \le 1$.*

Proof. This is clear by virtue of 2.10, 2.11, 2.12, 2.13 and 2.15.

Note that $q(S) = 0$ if S is either of type $(1, 2)$ or of type $(1, 1)$ (by Bombieri [2]). So $q(V) = 0$ by Kollár's formula.

If S is of type $(1, 1)$, then $f_*\omega_{X'}$ is a line bundle of positive degree because $|K_{X'}|$ is composed with a pencil and $p_g(X) > 1$. So one has, by Kollár's formula (Proposition 7.6 of [20]),

$$h^2(\mathcal{O}_V) = h^2(\mathcal{O}_X) = h^1(B, f_*\omega_{X'}) + h^0(B, R^1 f_*\omega_{X'}) = 0.$$

If S is of type $(1, 2)$, according to the result (the middle part) at page 524 of [6], we have $h^2(\mathcal{O}_V) \le 1$. We are done. $\square$

There is an example which shows that our estimation is sharp.

Example 2.19. In [9], Fletcher found a canonical 3-fold with $K^3 = \frac{1}{3}$ and $p_g = 2$ as a hypersurface $X_{16} \subset \mathbb{P}(1, 1, 2, 3, 8)$. The example has 3 terminal singularities of type $2 \times \frac{1}{2}(1, -1, 1)$, $1 \times \frac{1}{3}(1, -1, 1)$. This fits in with our argument above since this 3-fold has the canonical stability index 8, and the 7-canonical map is generically finite of degree 2.

Remark 2.20. Probably there are more examples. It would be very interesting to find new examples with $p_g = 2$ and $n_0 = 8$.

3. The case $p_g(V) \ge 3$ and the proof of Theorem 1.4

Assume from now on that $p_g(V) \ge 3$. Set $d := \dim \varphi_1(X)$.

3.1. The case $d = 3$. In this case, $p_g(X) \ge 4$. It is obvious that $K^3 \ge 2$. In fact, one has a general inequality $K^3 \ge 2p_g(X) - 6$ which may be obtained by induction on the dimension (see the main theorem in [19]). Anyway that is another kind of question.

3.2. The case $d = 2$. We have an induced fibration $f : X' \longrightarrow B$ from φ_1. By an easy subadditivity (1.7 in [23]), a general fibre C is a smooth curve of genus ≥ 2. We have $\pi^*(K_X) = S + E'$ and then

$$\pi^*(K_X)|_S = S|_S + E'|_S.$$

Because $d = 2$, $S|_S \equiv a_2 C$ where $a_2 \geq p_g(X) - 2$ and that the equality holds if and only $|S|_S|$ is a rational pencil. Thus

$$K^3 \geq \pi^*(K_X)^2 \cdot S \geq a_2 \pi^*(K_X) \cdot C.$$

We have $p = 1$ and may set $\beta = a_2 \geq 1$. Set $G := C$. We hope to run 2.4. In the proof of Case 2 of Theorem 3.1 at page 521 in [6], we have shown $\xi \geq \frac{6}{7}$. Take $m_1 = 8$. Then $\alpha = (m_1 - 1 - 1 - \frac{1}{\beta})\xi > 4$. This gives $\xi \geq \frac{7}{8}$. Similarly one may use induction to see $\xi \geq \frac{k}{k+1}$ for all $k > 8$. Taking limits, one gets $\xi \geq 1$.

So when $p_g(X) = 3$, we have $K_X^3 \geq 1$.

When $p_g(X) \geq 4$, we have $K_X^3 \geq 2$.

In fact, if $g(C) \geq 3$ and take $m = 6$, the technique 2.4 gives $\xi > 1$. Whenever the equalities for K^3 hold above, one must have $\xi = 1$. So we see that C must be a curve of genus 2, and that $|S|_S|$ must be a rational pencil (by Riemann-Roch) and so that B is a birationally ruled surface.

3.3. The case $d = 1$ and $b > 0$. In this case, one has $\pi^*(K_X) \equiv \bar{a}_2 S + E'$ where $\bar{a}_2 \geq p_g(X) \geq 3$. Also one has $\pi^*(K_X) \sim \sigma^*(K_{S_0})$ because one may take a X such that the movable part of $|K_X|$ is base point free by Lemma 2.3. One apparently has $K_X^3 \geq 3$. We are done.

Now we see how to find a bigger β than those found in the last section.

Lemma 3.4. *Assume $p_g(X) = p + 1 \geq 3$, $d = 1$ and $b = 0$. Then one may take a rational number $\beta_0 \mapsto \frac{p}{p+1}$ such that $\pi^*(K_X) - \beta_0 \sigma^*(K_{S_0})$ is pseudo-effective.*

Proof. Because $|K_X|$ is composed with a rational pencil and $P_g(X) = p + 1$, one has

$$\mathcal{O}_B(p) \hookrightarrow f_* \omega_{X'}.$$

Thus we have

$$f_* \omega_{X'/B}^{4p} \hookrightarrow f_* \omega_{X'}^{4p+8}.$$

For any integer k, denote by M_k the movable part of $|kK_{X'}|$. Note that $f_* \omega_{X'/B}^{4p}$ is generated by global sections (see [12, 16, 20, 24, 31]) and so that any local section can be extended to a global one. On the other hand, $|4p\sigma^*(K_{S_0})|$ is base point free and is exactly the movable part of $|4pK_S|$ by Bombieri [2] or Reider [28]. Applying Lemma 2.7 of [7], one has

$$(4p + 8)\pi^*(K_X)|_S \geq M_{4p+8}|_S \geq 4p\sigma^*(K_{S_0}).$$

This means that there is an effective $\mathbb{Q}$-divisor E_0' such that

$$(4p + 8)\pi^*(K_X)|_S =_{\mathbb{Q}} 4p\sigma^*(K_{S_0}) + E_0'.$$

Thus

$$\pi^*(K_X)|_S \equiv \frac{p}{p+2}\sigma^*(K_{S_0}) + E_0$$

where $E_0 = \frac{1}{4p+8}E_0'$ is an effective $\mathbb{Q}$-divisor. Set $a_0 := 4p + 8$ and $b_0 := 4p$.

Assume we have defined a_n and b_n. We describe a_{n+1} and b_{n+1} inductively such that $\beta_0 \geq \frac{b_{n+1}}{a_{n+1}}$. One may assume from the beginning that $a_n\pi^*(K_X)$ supports on a divisor with normal crossings. Then the Kawamata-Viehweg vanishing theorem implies the surjective map

$$H^0(K_{X'} + \ulcorner a_n\pi^*(K_X)\urcorner + S) \longrightarrow H^0(S, K_S + \ulcorner a_n\pi^*(K_X)\urcorner|_S).$$

That means

$$
\begin{aligned}
|K_{X'} + \ulcorner a_n\pi^*(K_X)\urcorner + S|\big|_S &= |K_S + \ulcorner a_n\pi^*(K_X)\urcorner|_S| \\
&\supset |K_S + b_n\sigma^*(K_{S_0})| \\
&\supset |(b_n + 1)\sigma^*(K_{S_0})|.
\end{aligned}
$$

Denote by M_{a_n+1}' the movable part of $|(a_n + 1)K_{X'} + S|$. Applying Lemma 2.7 of [7] again, one has

$$M_{a_n+1}'|_S \geq (b_n + 1)\sigma^*(K_{S_0}).$$

Re-modifying our original π such that $|M_{a_n+1}'|$ is base point free. In particular, M_{a_n+1}' is nef. According to 1.3, $|mK_X|$ gives a birational map whenever $m \geq 6$. Thus M_{a_n+1}' is big.

Now the Kawamata-Viehweg vanishing theorem gives

$$
\begin{aligned}
|K_{X'} + M_{a_n+1}' + S|\big|_S &= |K_S + M_{a_n+1}'|_S| \\
&\supset |K_S + (b_n + 1)\sigma^*(K_{S_0})| \\
&\supset |(b_n + 2)\sigma^*(K_{S_0})|.
\end{aligned}
$$

We may repeat the above procedure inductively. Denote by M_{a_n+t}' the movable part of $|K_{X'} + M_{a_n+t-1}' + S|$. For the same reason, we may assume $|M_{a_n+t}'|$ to be base point free. Inductively one has

$$M_{a_n+t}'|_S \geq (b_n + t)\sigma^*(K_{S_0}).$$

Applying the vanishing theorem once more, we have

$$
\begin{aligned}
|K_{X'} + M_{a_n+t}' + S|\big|_S &= |K_S + M_{a_n+t}'|_S| \\
&\supset |K_S + (b_n + t)\sigma^*(K_{S_0})| \\
&\supset |(b_n + t + 1)\sigma^*(K_{S_0})|.
\end{aligned}
$$

Take $t = p$. Noting that

$$|K_{X'} + M_{a_n+p}' + S| \subset |(a_n + p + 1)K_{X'}|$$

and applying Lemma 2.7 of [7] again, one has

$$(a_n + p + 1)\pi^*(K_X)|_S \geq M_{a_n+p+1}|_S \geq (b_n + p)\sigma^*(K_{S_0}).$$

Set $a_{n+1} := a_n + p + 1$ and $b_{n+1} = b_n + p$. We have seen

$$\beta_0 \geq \frac{b_{n+1}}{a_{n+1}}.$$

A direct calculation gives

$$a_n = n(p+1) + a_0$$
$$b_n = np + b_0.$$

Take the limit with $n \mapsto +\infty$, one has $\beta_0 \geq \frac{p}{p+1}$. We are done. □

3.5. The type 2^+**.** We have $K_X^3 \geq (p_g(X) - 1)\pi^*(K_X)^2 \cdot S$ by the equation (2.1). The main point is to estimate $\pi^*(K_X)^2 \cdot S$. We still set $G = 2\sigma^*(K_{S_0})$. We have $p \geq 2$ by definition. By 3.4, one may find a rational number $\beta = \frac{1}{2}\beta_0 \mapsto \frac{1}{3}$ whenever $p_g(X) = 3$ (or $\frac{3}{8}$ whenever $p_g(X) \geq 4$) such that $\pi^*(K_X)|_S - \beta G$ is pseudo-effective. Now we have

$$\xi = \pi^*(K_X)|_S \cdot G \geq \beta G^2 = 4K_{S_0}^2 \beta \geq 8\beta.$$

Thus we have $K^3 \geq (p_g(X) - 1)\beta\xi$.

  This gives $K^3 \geq \frac{16}{9}$ if $p_g(X) = 3$ and $K^3 \geq \frac{27}{8}$ if $p_g(X) \geq 4$.

3.6. The type $(2, 3)$**.** Let C be a generic irreducible element of $|G|$. Then C is a curve of genus 3 as we know. It has been shown in 2.12 that $\xi \geq 1$. On the other hand, 3.4 tells that one may take a rational number $\beta \mapsto \frac{2}{3}$ whenever $p_g(X) = 3$ (or $\frac{3}{4}$ whenever $p_g(X) \geq 4$).

  Consider the case $p_g(X) = 3$. Take $m_1 = 5$ and we have $p = 2$. Then $\alpha = (m_1 - 1 - \frac{1}{2} - \frac{1}{\beta})\xi \geq 2$. Then one has $\xi \geq \frac{6}{5}$. Take $m_2 = 5$ again. One sees that $\alpha > 2$. This means that φ_5 is birational. Take $m_3 = 4$. Then $\alpha = (4 - 1 - \frac{1}{2} - \frac{1}{\beta})\xi > 1$. One has $\xi \geq \frac{3}{2}$. It seems that this is the best one can get under the condition $p_g(X) = 3$. Thus $K^3 \geq 2\beta\xi \geq 2$.

  Whenever $p_g(X) \geq 4$, one has $K^3 \geq 3$. There might be better bound.

3.7. The type $(1, 2)$**.** This is a continuation of 2.15 under the assumption $p_g(X) \geq 3$. We have got $\xi \geq \frac{2}{3}$ in 2.15 because the proof for $p_g(X) = 2$ is automatically true for $p_g(X) \geq 3$. Recall that G is the movable part of $|\sigma^*(K_{S_0})|$. We have $p = 2$ and $\beta = \beta_0$ is near $\frac{2}{3}$ by 3.4. Take $m = 5$. Then $\alpha = (5 - 1 - \frac{1}{2} - \frac{1}{\beta})\xi > 1$. This gives $\xi \geq \frac{4}{5}$. Take $m = 6$. Then it gives $\xi \geq \frac{5}{6}$. An induction step gives $\xi \geq 1$.

  Now if $p_g(X) = 3$, then $K^3 \geq 2\beta\xi \geq \frac{4}{3}$.

  If $p_g(X) = 4$, then $K_X^3 \geq 3\beta\xi \geq \frac{9}{4}$.

3.8. The type $(1, 1)$**.** Comparing with 2.13, we have better situation because $p_g(X) \geq 3$. We keep the same notations and pace as in 2.13.

  We still consider the natural restriction map γ:

$$H^0(X', M_2) \xrightarrow{\gamma} U_2 \subset H^0(S, M_2|_S) \subset H^0(S, 2K_S),$$

where U_2 is the image of γ as a $\mathbb{C}$-subspace of $H^0(S, M_2|_S)$. If $p_g(X) = 3$, then a similar calculation to that in 2.13 gives $\chi(\mathcal{O}_X) = -2$ and then the plurigenus formula of Reid gives $P_2(X) \geq 7$. If $p_g(X) \geq 4$, then $\chi(\mathcal{O}_X) \leq -3$ and $P_2(X) \geq 10$.

Case 1. $\dim_{\mathbb{C}} U_2 = 3$.

All those arguments may be copied from Case 1 of 2.13. We only write down the estimation. We have

$$2\pi^*(K_X)|_{S_2} \geq S_2|_{S_2} = L_2.$$

By direct calculation on the surface S_2, we have

$$L_2^2 \geq 4(\dim_{\mathbb{C}} \overline{\Lambda_2} - 1) \geq 4(P_2(X) - 3).$$

Therefore we have

$$K_X^3 \geq \frac{1}{8}L_2^2 \geq \frac{1}{2}(P_2(X) - 3).$$

If $p_g(X) = 3$, then $K^3 \geq 2$.
If $p_g(X) \geq 4$, then $K^3 \geq \frac{7}{2}$.

Case 2. $\dim_{\mathbb{C}} U_2 = 2$.

We may copy the proof in Case 2 of 2.13. In this case, φ_2 induces a fibration $f_2 : X' \longrightarrow B_2$ with general fibre a smooth curve C. Lemma 3.4 allows us to choose a rational number $\beta_0 \mapsto \frac{2}{3}$ when $p_g(X) = 3$ (or $\frac{3}{4}$ when $p_g(X) \geq 4$) such that $\pi^*(K)|_S - \beta_0\sigma^*(K_{S_0})$ is pseudo-effective. Thus

$$(\pi^*(K_X) \cdot C)_{X'} = \pi^*(K_X)|_S \cdot C \geq \beta_0\sigma^*(K_{S_0}) \cdot C \geq 2\beta_0$$

where Claim 2.14 is applied once more to get the inequality. Recall that S_2 is a general member of the movable part of $|2K_{X'}|$. We may write

$$S_2|_{S_2} \sim \sum_{i=1}^{a_2} C_i \equiv a_2 C,$$

where the $C_i's$ are fibers of f_2 and $a_2 \geq P_2(X) - 2$. Noting that $2\pi^*(K_X)|_{S_2} \geq S_2|_{S_2}$, we get

$$
\begin{aligned}
4K_X^3 \geq 2\pi^*(K_X)^2 \cdot S_2 &= 2(\pi^*(K_X)|_{S_2})_{S_2}^2 \\
&\geq (P_2 - 2)(\pi^*(K_X)|_{S_2} \cdot C)_{S_2} \\
&= (P_2 - 2)(\pi^*(K_X) \cdot C)_{X'} \geq 2(P_2 - 2)\beta_0.
\end{aligned}
$$

If $p_g(X) = 3$, then $K^3 \geq \frac{5}{3}$.
If $p_g(X) \geq 4$, then $K^3 \geq 3$.

Case 3. $\dim_{\mathbb{C}} U_2 = 1$.
Similar to the proof in Case 3 of 2.13, we have

$$2\pi^*(K_X) \equiv a'S + E'$$

where $a' \geq P_2(X) - 1$. So we get

$$2K_X^3 \geq (P_2 - 1)(\pi^*(K_X)|_S)_S^2 \geq (P_2 - 1)\beta_0 \pi^*(K_X)|_S \cdot \sigma^*(K_{S_0})$$

where $\beta_0 \mapsto \frac{2}{3}$ when $p_g(X) = 3$ (or $\frac{3}{4}$ otherwise) by Lemma 3.4.

Now we have to estimate $\pi^*(K_X)|_S \cdot \sigma^*(K_{S_0})$. We have $p = 2$ and set $G := 2\sigma^*(K_{S_0})$ and $\beta = \frac{1}{2}\beta_0$. Let C be a generic irreducible element of $|G|$. Then C is non-hyperelliptic as we have seen and $\deg(K_C) = 6$. Take $m_1 = 6$. Then $\alpha = (m_1 - 1 - \frac{1}{2} - \frac{1}{\beta})\xi > 0$. The even divisor G gives $\alpha_0 \geq 2$. 2.4 gives $\xi \geq \frac{4}{3}$. This might be the best that we get.

We have $K^3 \geq \frac{1}{2}(P_2 - 1)\beta\xi$.

If $p_g(X) = 3$, then $K^3 \geq \frac{4}{3}$.

If $p_g(X) \geq 4$, then $K^3 \geq \frac{9}{4}$. We are done.

Corollary 3.9. *Let V be a smooth projective 3-fold of general type. Then*

(1) $K^3 \geq 1$ whenever $p_g(V) = 3$;

(2) $K^3 \geq 2$ whenever $p_g(V) = 4$.

Proof. This is clear by 3.1, 3.2, 3.3, 3.5, 3.6, 3.7 and 3.8. $\square$

3.10. Proof of Theorem 1.4.

Proof. 2.10, 2.11, 2.12, 2.13, 2.15, Corollary 3.9 and Example 2.19 imply Theorem 1.4(1).

Corollary 2.17 and Corollary 3.9 imply Theorem 1.4(2). $\square$

3.11. Proof of Theorem 1.5.

Proof. Corollary 3.9 gives the inequalities in (1) and (2). Examples 3.12, 3.13 shows that both the lower bounds are sharp.

Theorem 1.5 (3) is due to Corollary 2.18.

Theorem 1.5 (4) and (5) are due to 3.1 through 3.8. $\square$

Example 3.12. Fletcher found an example in [9] with $K^3 = 1$ and $p_g(X) = 3$. This example of canonical 3-fold is a hypersurface : $X_{12} \subset \mathbb{P}(1, 1, 1, 2, 6)$ with terminal singularities. This example has 2 singularities of type $\frac{1}{2}(1, -1, 1)$. One may see that X is canonically fibred by curves of genus 2 and $n_0(X) = 6$. The 5-canonical map is generically finite of degree 2.

Example 3.13. There is a trivial example of a 3-fold of general type with $K^3 = 2$ and $p_g = 4$. Take a double cover over $\mathbb{P}^4$ with branch locus a smooth hypersurface of degree 12. Then one gets a smooth canonical 3-fold with $K^3 = 2$ and $p_g = 4$ and the canonical map is finite of degree 2 and $n_0(X) = 5$.

4. An application to certain 4-folds

4.1. Let Y be a minimal normal projective 4-fold of general type with only canonical singularities. Assume $p_g(Y) \geq 2$. Pick up a sub-pencil

$\Lambda \subset |K_Y|$ with $\dim \Lambda = 1$. Take birational modifications $\pi : Y' \longrightarrow Y$ like in 2.1 such that the movable part of $\pi^*(\Lambda)$ is base point free. After taking the Stein factorization, we get an induced fibration $f : Y' \longrightarrow B$ where B is a smooth curve. Denote by $|M|$ the movable part of $\pi^*(\Lambda)$ and by X a generic irreducible element of $|M|$. Naturally X would be a smooth projective 3-fold of general type by a weak subadditivity (see (1.7) in [23]).

One might hope to know whether there are parallel results to [6] in 4-dimensional case. This is still open because of a pathological case. For example, take Y to be a product $X \times C$ where X is a minimal 3-fold of general type with $p_g(X) = 1$ and C a smooth curve of genus $g \geq 2$. One would see that one knows nothing to Y since little is known to X. So, here, we will exclude this case.

4.2. Proof of Theorem 1.6.

Proof. We consider the natural map

$$H^0(Y', K_{Y'}) \xrightarrow{\theta} U \subset H^0(X, K_{Y'}|_X) \subset H^0(X, K_X)$$

where U is the image of θ. If $\dim \Phi_{K_Y}(Y) \geq 2$, then $\dim_{\mathbb{C}}(U) \geq 2$ and so $p_g(X) \geq 2$. If $|K_Y|$ is composed with a pencil of 3-folds X, by the assumption of Theorem 1.6, we still have $p_g(X) \geq 2$. Therefore we have $n_0(X) \leq 8$ according to [6].

1) Set $g := \Phi_\Lambda \circ \pi$. Then $g : Y' \longrightarrow \mathbb{P}^1$ is a surjective morphism. One sees that a general fiber of g is a smooth projective 3-dimensional scheme with each component a 3-fold of general type. We apply Kollár's technique ([20]) to study the canonical stability index. We have

$$\mathcal{O}_{\mathbb{P}^1}(1) \hookrightarrow g_*\omega_{Y'}$$

and so

$$g_*\omega_{Y'/\mathbb{P}^1}^8 \hookrightarrow g_*\omega_{Y'}^{24}.$$

The standard semi-positivity says that $g_*\omega_{Y'/\mathbb{P}^1}^8$ is generated by global sections. This means that any local section of $g_*\omega_{Y'/\mathbb{P}^1}^8$ can be glued to a global section of $g_*\omega_{Y'}^{24}$. On the other hand, one may apply the Kawamata-Viehweg vanishing theorem to see that $|24K_{Y'}|$ may separate different generic irreducible elements of $|M|$. Thus the Tankeev principle gives $n_0(Y) \leq 24$.

2) We study the induced fibration $f : Y' \longrightarrow B$ which is the Stein factorization of g. We may write

$$\pi^*(K_Y) \sim M + E' \equiv a_4 X + E'$$

where E' is an effective $\mathbb{Q}$-divisor and $a_4 > b = g(C)$.

The assumption implies $b = 0$. We first pick up a minimal model X_0 of X and denote by $\sigma : X \longrightarrow X_0$ the "contraction" onto the minimal model. Denote by r the canonical index of X_0. Then, according to

the Base Point Free Theorem (see Chapter 3 in [18]), $|mrK_{X_0}|$ is base point free for larger m. We have an inclusion

$$f_*\omega_{Y''/\mathbb{P}^1}^{mr} \hookrightarrow f_*\omega_{Y'}^{3mr}.$$

Since $f_*\omega_{Y''/\mathbb{P}^1}^{mr}$ is generated by global sections and the movable part of $|mrK_X|$ is $mr\sigma^*(K_{X_0})$, Lemma 2.7 of [7] gives

$$3mr\pi^*(K_Y)|_X \geq mr\sigma^*(K_{X_0})$$

for any larger m. Therefore we may find a rational number $\beta \mapsto \frac{1}{3}$ such that $\pi^*(K_Y)|_X - \beta\sigma^*(K_{X_0})$ is pseudo-effective. Thus we get

$$K_Y^4 \geq \pi^*(K_Y)^3 \cdot X \geq \frac{1}{27}K_{X_0}^3 \geq \frac{1}{81}$$

by Theorem 1.4. We are done.   □

Remark 4.3. If there is a sound relatively minimal model program like in 3-dimensional case, then a parallel Lemma to Lemma 2.3 exists. So in the proof above we do not need to avoid the case $b > 0$. Then Theorem 1.6 can be strengthened to the following form:

> Let Y be a minimal (i.e. K_Y being nef) projective 4-fold of general type with only canonical singularities. Assume $p_g(Y) \geq 2$ and Y is not canonically fibred by 3-folds of geometric genus 1. Then
> (1) $n_0(Y) \leq 24$;
> (2) $K_Y^4 \geq \frac{1}{81}$.

References

[1] W. Barth, C. Peter, A. Van de Ven, *Compact complex surface*, Springer-Verlag, 1984.

[2] E. Bombieri, *Canonical models of surfaces of general type*, Publications I.H.E.S. **42**(1973), 171-219.

[3] F. Catanese, *Surfaces with $K^2 = p_g = 1$ and their period mapping*, Springer Lecture Notes in Math. **732**(1979), 1-29.

[4] F. Catanese, *Pluricanonical mappings of surfaces with $K^2 = 1, 2$, $q = p_g = 0$*, in 'C.I.M.E. 1977: Algebraic surfaces', Liguori, Napoli (1981), 247-266

[5] F. Catanese, C. Ciliberto, *Surfaces with $p_g = q = 1$*. Problems in the theory of surfaces and their classification (Cortona, 1988), 49-79, Sympos. Math., XXXII, Academic Press, London, 1991.

[6] M. Chen, *Canonical stability of 3-folds of general type with $p_g \geq 3$*, International J. Math. **14**(2003), 515-528.

[7] M. Chen, *Canonical stability in terms of singularity index for algebraic 3-folds*, Math. Proc. Cambridge Phil. Soc. **131**(2001), 241-264.

[8] C. Ciliberto, *The bicanonical map for surfaces of general type*, Proc. Symposia in Pure Math. **62**(1997), 57-83.

[9] A. Corti, M. Reid, *Explicit birational geometry of 3-folds*. London Mathematical Society, Lecture Note Series, 281. Cambridge University Press, Cambridge, 2000.

[10] H. Esnault, E. Viehweg, *Lectures on Vanishing Theorems*. DMV-Seminar **20**(1992), Birkhäuser, Basel-Boston-Berlin.

[11] P. Francia, *On the base points of the bicanonical system*. Problems in the theory of surfaces and their classification (Cortona, 1988), 141–150, Sympos. Math., XXXII, Academic Press, London, 1991.

[12] T. Fujita, *On Kahler fiber spaces over curves*, J. Math. Soc. Japan **30** (1978), 779-794.

[13] Christopher Hacon, A private letter to the author.

[14] Christopher D. Hacon and James Mckernan, *Boundedness of pluricanonical maps of varieties of general type*, preprint, math.AG/0504327

[15] Y. Kawamata, *A generalization of Kodaira-Ramanujam's vanishing theorem*, Math. Ann. **261**(1982), 43-46.

[16] Y. Kawamata, *Kodaira dimension of algebraic fiber space over curves*, Invent. Math. **66**(1982), 57-71.

[17] Y. Kawamata, *The Zariski decomposition of log-canonical divisors*. Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 425–433, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[18] Y. Kawamata, K. Matsuda, K. Matsuki, *Introduction to the minimal model problem*, Adv. Stud. Pure Math. **10**(1987), 283-360.

[19] M. Kobayashi, *On Noether's inequality for threefolds*. J. Math. Soc. Japan 44 (1992), no. 1, 145-156.

[20] J. Kollár, *Higher direct images of dualizing sheaves*, I, Ann. of Math. **123**(1986), 11-42; II, ibid. **124**(1986), 171-202.

[21] J. Kollár, S. Mori, Birational geometry of algebraic varieties, 1998, Cambridge Univ. Press.

[22] Y. Miyaoka, *Tricanonical maps of numerical Godeaux surfaces*. Invent. Math. **34**(1976), 99-111.

[23] S. Mori, *Classification of higher-dimensional varieties*. Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 269-331, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[24] N. Nakayama, *Hodge filtrations and the higher direct images of canonical sheaves*, Invent. Math. **85**(1986), 237-251.

[25] K. Ohno, *Some inequalities for minimal fibrations of surfaces of general type over curves*, J. Math. Soc. Japan **44**(1992), 643-666.

[26] M. Reid, *Young person's guide to canonical singularities*, Proc. Symposia in pure Math. **46**(1987), 345-414.

[27] M. Reid, *Minimal models of canonical 3-folds*, Adv. Stud. Pure Math. **1**(1983), 131-180.

[28] I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. Math. **127**(1988), 309-316.

[29] H. Tsuji, *Pluricanonical systems of projective 3-folds of general type*, preprint, math.AG/0204096.

[30] E. Viehweg, *Vanishing theorems*, J. reine angew. Math. **335**(1982), 1-8.

[31] E. Viehweg, *Weak positivity and the additivity of the Kodaira dimension of certain fibre spaces*, Adv. Studies in Pure Math. **1**(1983), 329-353.

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